SELF-SHRINKERS TO THE MEAN CURVATURE FLOW ASYMPTOTIC TO ISOPARAMETRIC CONES

by
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Abstract

We record work done by the author joint with Professor Joel Spruck[9] on constructing self-shrinking end to the mean curvature flow asymptotic to an isoparametric cone C and lying outside of C. We call a cone C in $\mathbb{R}^{n+1}$ an isoparametric cone if C is the cone over a compact embedded isoparametric hypersurface $\Gamma \subset \mathbb{S}^n$. The theory of isoparametric hypersurface is extremely rich and there are infinitely many distinct classes of examples, each with infinitely many members.

READERS: Professor Joel Spruck, Professor Yingying Zhang.

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Introduction

A hypersurface Σ in R^{n+1} is said to be a self-shrinker for the mean curvature flow if $\Sigma_t = \sqrt{-t}\Sigma$ flows by homothety starting at $t = -1$ until it disappears at time $t = 0$. A simple computation shows that Σ_t is a self-shrinker if and only if Σ satisfies the equation

$$H = -\frac{1}{2} X \cdot \nu, \tag{1.0.1}$$

where H is the mean curvature, X is the position vector, and ν is the unit normal of Σ. The study of self-similar shrinking solutions to the mean curvature flow is now well understood to be an important and essential feature of the classification of possible singularities that may develop. In fact, the monotonicity formula of Huisken[18, 17] and a rescaling argument of Ilmanen and White imply that suitable blowups of singularities of the mean curvature flow are self-shrinkers[12]. By a theorem of Wang[31], if C is a smooth regular cone with vertex at the origin, there is at most one self-shrinker with an end asymptotic to C.

There are relatively few constructions in the literature of self-shrinkers asymptotic to a cone C, for example, Kleene and Moller[21] have constructed examples in rotational symmetric case and classified all possible cases. In this paper, we will construct infinitely many families of special mean convex cones C with interesting topology and a corresponding end of a self-shrinker which is asymptotic to C and lies outside of the cone. A closed connected compact hypersurface $\Gamma \subset S^n \subset R^{n+1}$ is called an isoparametric hypersurface if its principal curvatures are constant. Equivalently, Γ is part of a family of parallel hypersurfaces in S^n which have constant mean curvature. By a theorem of Cecil and Ryan[8], Γ is taut and so is automatically embedded. We will say that a cone C in R^n is an isoparametric cone if C is the cone over an embedded isoparametric hypersurface Γ.

The theory of isoparametric hypersurfaces in S^n is extremely rich and beautiful. Cartan classified all isoparametric hypersurfaces in S^n with $g \leq 3$ distinct principal curvatures. For $g = 1$, they are totally umbilic hyperspheres, while for $g = 2$, they are standard product of spheres $S^p(a) \times S^q(b)$, where $a^2 + b^2 = 1$ and $p + q + 1 = n$. For $g = 3$, Cartan showed that all the principal curvatures have the same multiplicity $m = 1, 2, 4, 8$ and Γ must be a tube of constant radius over a standard embedding of a projective plane $\mathbb{F}P^2$ into S^{3m+1}, where $\mathbb{F}$ is the division algebra $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. In the process of proving this result, he showed that any such Γ with g distinct principal curvatures of the same multiplicity can be defined by the restriction to S^n if a homogeneous harmonic polynomial F of degree g on R^{n+1} satisfying $|\nabla F|^2 = g^2|s|^{2g-2}$. Munzner found a remarkable structural generalization of this last result of Cartan. Let $\Gamma \subset S^n$ be an isoparametric hypersurface with g distinct principal curvatures. Then there is a homogeneous polynomial F of degree g defined in all of R^{n+1}, Cartan-Munzner polynomial, satisfying $|\nabla F|^2 = g^2|x|^{2g-2}$, $\triangle F = \frac{m_- - m_+}{2} g^2 |x|^{g-2}$, where m_-, m_+ are the multiplicity of these g distict principal curvatures. The restriction f of F to S^n has range $[-1, 1]$ and satisfies

$$|\nabla f|^2 = g^2(1 - f^2), \triangle_{S^n} f + g(n + g - 1)f = \frac{m_- - m_+}{2} g^2.$$

Each member Γ_t of the isoparametric family determined by Γ has the same focal sets $M_\pm := f^{-1}(\pm 1)$ which are smooth minimal submanifolds of codimension $m_\pm + 1$ respectively. In proving this result, Munzner shows that if the principal curvatures of Γ are written as $\cot \theta_k, 0 < \theta_1 < \cdots < \theta_g < \pi$, then $\theta_k = \theta_1 + \frac{k-1}{g}\pi$ with multiplicities $m_k = m_{k+2}$ subscrips mod g. Thus for g odd, all multiplicities are the same while for g even, there are at most two distinct multiplicities m_-, m_+. Moreover each Γ_t separates $\mathbb{S}^n$ into two connected components $D_\pm$ such that $D_\pm$ is a disk bundle with fibres of dimension $m_\pm + 1$ over $M_\pm$. From this he is able to deduce using algebraic topology that $g \in \{1, 2, 3, 4, 6\}$. Earlier, Takagi and Takahashi[30] had classified all homogeneous examples based on the work of Hsiang and Lawson[16].

For $g = 4$ using representations of Clifford algebras, Ozeki and Takeuchi[28, 29] found two classes of examples, each with infinitely many members of inhomogeneous solutions. Later these methods were greatly generalized by Ferus, Karcher, and Munzner[15], who showed there are infinitely many distinct classes of solutions in odd dimensional spheres $\mathbb{S}^{2l-1} \subset \mathbb{R}^{2l}$ each with infinitely many members. Their examples contain almost all known homogeneous and inhomogeneous examples. For $g = 6$, Abresch[1] showed that $m_- = m_+ = 1$ or 2, so examples occur only in dimension $n = 7, 13$.

From the isoparametric function f, we easily compute the mean curvature $H(f)$ of the level set

$f = t$:

$$H(f) = \frac{1}{|\nabla_{\mathbb{S}^n} f|} \triangle_{\mathbb{S}^n} f = \frac{1}{\sqrt{1-f^2}} \{g(\frac{m_- - m_+}{2}) - (n+g-1)f\},$$

and so the level set $f = \frac{g(m_- m_+)}{2(n+g-1)}$ is the unique minimal isoparametric hypersurface of the family. The main theorem in this thesis is stated as follows.

Theorem 1. *Let $\Gamma \subset \mathbb{S}^n$ be an embedded isoparametric hypersurface and C be the cone over Γ. Then there is a radial graph $S = \{e^{v(d(z))} : z \in A\}$, where $d(z)$ is the distance function to Γ, $A = \{z : 0 < d(z) \leq d_0 + \epsilon\}$, which is an end of a self-shrinker to the mean curvature flow and is asymptotic to C. Here the parallel hypersurface $d(z) = d_0$ is the unique minimal hypersurface of the family.*

1.1 Notation

Throughout the paper, C_k^n will be denoted by the combinatoric number $\frac{n!}{k!(n-k)!}$.

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Background

2.1 Mean Curvature Flow

A smooth family of manifolds $M_t^n \subset \mathbb{R}^{n+1}$ parametrized by $F : I \times M \to \mathbb{R}^{n+1}$ for some interval I is said to be moving by mean curvature flow if

$$\frac{\partial F}{\partial t} = -H\nu,$$

where ν is the outer normal of $M_t^n \subset \mathbb{R}^{n+1}$. Notice here, the mean curvature flow we introduce is an parametric model, which preserves its topology thoughout the existence. For non-parametric model, there are level-set mean curvature flow or Brakke flow. But the price we pay to fix the topology makes the curvature blow-up as approaching to the singular time[18, 17].

On the other hand, one can use maximum principle of heat operator to prove that if the initial manifold M is compact, then the mean curvature flow develops finite time singularity. For instance, if M is the round sphere, then it shrinks into the origin by $T = \frac{R^2}{2n}$.

Among all kinds of examples people discovered so far, there's one particular class of solutions called self-similar solution. Literally, suppose $I = [-T, 0)$ for convenience, and 0 is the singular time, the solution satisfies $M_t = \sqrt{-t}M$ is called a self-shrinker. It's not hard to derive if M is a self-shrinker, then it satisfies the equation

$$H = \frac{1}{2} < F_{-T}, \nu > .$$

For examples, the hyperplanes, the spheres, and the cylinders in $\mathbb{R}^{n+1}$ are intuitively be self-shrinkers.

The importance of self-similar solutions was first discovered by Huisken[17]. He defined the **Type I singularity** to be

$$\sup_{M_t} \|A\|^2 \leq \frac{C}{-t},$$

where A is the second fundamental form. It is shown that the behavior of a solution near type I singularity is asymptotic to a self-shinker. Hence, it's an interesting question to ask how many different types of shrinkers exist.

So far, to the author's knowledge, besides the flat planes, the round spheres, and the round cylinders; Angenent[2] made a strange-looking torus which is not fully understanding, for instance, we are not sure wheather the Angenent's torus is unique; Kapouleas, Kleene, and Moller[20] use glueing techniques to construct a class of examples of shrinkers of g genus when g is large; and Nguyen[25, 26, 27] also has some examples. The reason why it's hard to find self-shrinkers is that it's an unstable critical points to the entropy functional.[11, 10]

In order to understand the shrinkers better, we may want to detect the properties of shrinkers. One important result is shown by Wang[31] that if a solution of self-shrinkers that is asymptotic to a given regular cone exists, then it must be unique.

Definition 1. *Given a cone $C = \{\lambda\Gamma | \lambda \geq 0\}$, where Γ is a $n-1$ dimension surface, a **self-shrinking end** is defined to be the solution of self-shinker equation out side a compact region that is asymptotic to the cone C.*

Wang's result guarantees there is only one such solution if it exists, therefore it's well-defined. Later, Kleene and Moller[21] prove that given any $S^{n-1} \subset \mathbb{R}^n$, there is a self-shrinker out side a compact region asymptotic to the cone passing through this S^{n-1} with a tip at the origin, therefore, the existence of self-shrinking ends of the standard cones.

2.2 Isoparametric Surfaces in Sphere

Definition 2. *A smooth hypersurface M^{n-1} in $S^n \subset R^{n+1}$ is said to be an isoparametric surface if all of its principal curvatures are constants (not necessarily equal).*

An equivalent definition is to consider M as a level set of an isoparametric function f where the gradient and the Laplacian are in terms of the function itself, $|\nabla_{\mathbb{S}^n} f| = A(f), \triangle_{\mathbb{S}^n} f = B(f)$. For example, a family of circles $S^n \cap \{x_{n+1} = c\}$, where $-1 < c < 1$, is a family of isoparametric surfaces with all the principal curvatures are the same constant depending on c.

One important fact of this example is that not only the equator of the sphere is an isoparametric hypersurface but all the parallel surfaces are indeed isoparametric. In general, Cartan[3, 4, 5, 6] proved that isoparametric hypersurfaces must come as a family of parallel hypersurfaces. Thus, we can consider a region on the sphere which is consist of isoparametric hypersurfaces and parametrized by the distance function.

Another great contribution to this field is Münzner's work[23, 24] that gives some structural characterizations. Münzner first showed that the principal curvatures of an isoparametric hypersurface can have at most two different multiplicities. Specifically, he showed that, if we write the g distinct principal curvatures as $\cot\theta_k$, $0 < \theta_1 < \cdots < \theta_g < \pi$, then $\theta_k = \theta_1 + \frac{k-1}{g}\pi$ and the multiplicities $m_k = m_{k+2}$. Hence for g odd, all the multiplicities are equal while for g even, there are at most two multiplicities $m_\pm$.

Next he showed that the isoparametric function f defining M satiesfies the Münzner's equations

$$|\nabla_{\mathbb{R}^{n+1}} f|^2 = g^2 r^{2g-2}, \triangle_{\mathbb{R}^n+1} f = \frac{m_- - m_+}{2} g^2 r^{g-2},$$

and has to be a polynomial, which is called Cartan polynomial. Then Münzner first proved that the two focal manifolds have ball bundle structures and the cohomology rings can be calculated so that the restriction on g can only be $1, 2, 3, 4, 6$.

Before Münzner's results, Cartan was able to classify the isoparametric hypersurfaces in $\mathbb{S}^n$ for $g = 1, 2, 3$. When $g = 4$, Ferus, Karcher, and Münzner[15] used Clifford algebra to construct an infinite class of examples. To be noticed here, their examples did not have restrictions on space dimentions and these are the only known examples so far. Recently, $g = 6$ have been settled by the work of Münzner, Dorfmeister and Neher[14, 13], Abresch[1], and Miyaoki[22]. Our self-shrinking ends will be based on these examples, and for further details, the author suggests the book written by Cecil and Ryan[8], and Cecil[7].

2.3 Self-Shrinking Ends on Isoparametric Cones

Definition 3. *A cone $C = \{\lambda\Gamma : \lambda \geq 0\}$ is called an isoparametric cone if $\Gamma^{n-1} \subset \mathbb{S}^n$ is an isoparametric hypersurface.*

In order to give Kleene-Moller's result a new interpretation, we know from last section that hyper-spheres in $\mathbb{S}^n$ form an isoparametric family. Therefore, one may regard it as a special case of a self-shrinking end of an isoparametic cone C with Γ to be a hyper-sphere in the upper hemi-sphere.

As a consequence, one may ask is the existence of self-shrinking ends still true when we consider the problem over an isoparametric cone.

Suppose $z : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is an embedding of the unit sphere $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$ with Riemannian metric

$$\sigma_{ij} =< \partial_i z, \partial_j z >,$$

the Levi-Civita connection $\nabla = \nabla_{\mathbb{S}^n}$ and the second fundamental form of $\mathbb{S}^n$

$$k_{ij} = - < \partial_i \partial_j z, \nu >=< \partial_j z, \partial_i \nu >= \sigma_{ij},$$

where $\{\partial_i\}$ is the local coordinates, $<,>$ is the usual inner product in $\mathbb{R}^{n+1}$, and ν is the unit normal which is equal to the position vector z. In this section, we first derive the self-shrinker equation as graph over a region A on $\mathbb{S}^n$.

Let $M = \{e^{v(z)} z : z \in A\}$ to be a self-shrinker, where A is a region on S^n and $X = e^{v(z)} z$ denotes the position vector. Hence, by $< z, z_i >= 0$, we have the Riemannian metric on X to be

$$g_{ij} =< X_i, X_j >=< e^v v_i z + e^v z_i, e^v v_j z + e^v z_j >= e^{2v}(v_i v_j + \sigma_{ij}).$$

As a consequence,

$$g^{ij} = e^{-2v}(\sigma^{ij} - \frac{v^i v^j}{1 + |\nabla v|^2}).$$

Now, consider the vector $z - \sigma^{kl} v_k z_l$ on each X_i-direction,

$$\begin{aligned} < z - \sigma^{kl} v_k z_l, X_i > &=< z - \sigma^{kl} v_k z_l, e^v v_i z + e^v z_i > \\ &=< z, e^v v_i z > - \sigma^{kl} v_k < z_l, e^v z_i > \\ &= e^v v_i - \sigma^{kl} v_k e^v \sigma_{li} = 0. \end{aligned}$$

Thus, $z - \sigma^{kl} v_k z_l$ is in the normal direction to M and the unit normal along this direction is given by

$$\begin{aligned}
\nu_X &= \frac{z - \sigma^{kl} v_k z_l}{\sqrt{< z - \sigma^{kl} v_k z_l, z - \sigma^{ij} v_i z_j >}} \\
&= \frac{z - \sigma^{kl} v_k z_l}{\sqrt{< z, z > + \sigma^{kl} v_k \sigma^{ij} v_i < z_l, z_j >}} \\
&= \frac{z - \sigma^{kl} v_k z_l}{\sqrt{1 + |\nabla v|^2}}.
\end{aligned}$$

As a result,

$$\frac{1}{2} < X, \nu_X > = \frac{e^v}{2\sqrt{1 + |\nabla v|^2}}.$$

On the other hand,

$$\begin{aligned}
X_{ij} &= e^v \partial_j (v_i z + z_i) + e^v v_j (v_i z + z_i) \\
&= e^v (v_{ij} + v_i v_j) z + e^v (v_i z_j + v_j z_i) + e^v z_{ij}.
\end{aligned}$$

Therefore, the second fundamental form of M

$$\begin{aligned}
h_{ij} &= - < X_{ij}, \nu_X > \\
&= - < e^v (v_{ij} + v_i v_j) z + e^v (v_i z_j + v_j z_i) + e^v z_{ij}, \frac{z - \sigma^{kl} v_k z_l}{\sqrt{1 + |\nabla v|^2}} > \\
&= \frac{-e^v}{\sqrt{1 + |\nabla v|^2}} < (v_{ij} + v_i v_j) z + (v_i z_j + v_j z_i) + z_{ij}, z - v^l z_l > \\
&= \frac{-e^v}{\sqrt{1 + |\nabla v|^2}} [(v_{ij} + v_i v_j) + < z_{ij}, z > - (v_i v^l \sigma_{jl} + v_j v^l \sigma_{il}) - v^l < z_{ij}, z_l >] \\
&= \frac{-e^v}{\sqrt{1 + |\nabla v|^2}} [(v_{ij} - \sigma^{kl} < z_{ij}, z_l > v_k) - v_i v_j - k_{ij}] \\
&= \frac{-e^v}{\sqrt{1 + |\nabla v|^2}} (\nabla_i \nabla_j v - v_i v_j - \sigma_{ij}).
\end{aligned}$$

As a consequence, the mean curvature of M with respect to $(\mathbb{S}^n, \sigma)$

$$\begin{aligned}
H = g^{ij} h_{ij} &= e^{-2v} (\sigma^{ij} - \frac{v^i v^j}{1 + |\nabla v|^2}) \cdot \frac{-e^v}{\sqrt{1 + |\nabla v|^2}} (\nabla_i \nabla_j v - v_i v_j - \sigma_{ij}) \\
&= \frac{-e^{-v}}{\sqrt{1 + |\nabla v|^2}} [(\sigma^{ij} - \frac{v^i v^j}{1 + |\nabla v|^2}) \nabla_i \nabla_j v - \sigma^{ij} v_i v_j - \sigma^{ij} \sigma_{ij} + \frac{v^i v^j v_i v_j}{1 + |\nabla v|^2} + \frac{\sigma_{ij} v^i v^j}{1 + |\nabla v|^2}] \\
&= \frac{-e^{-v}}{\sqrt{1 + |\nabla v|^2}} [(\sigma^{ij} - \frac{v^i v^j}{1 + |\nabla v|^2}) \nabla_i \nabla_j v - n].
\end{aligned}$$

Finally, by the self-shrinker equation $H = \frac{1}{2} < X, \nu_X >$,

$$\frac{-e^{-v}}{\sqrt{1+|\nabla v|^2}}[(\sigma^{ij} - \frac{v^i v^j}{1+|\nabla v|^2})\nabla_i \nabla_j v - n] = \frac{e^v}{2\sqrt{1+|\nabla v|^2}},$$

we obtain

$$(\sigma^{ij} - \frac{v^i v^j}{1+|\nabla v|^2})\nabla_i \nabla_j v = n - \frac{1}{2}e^{2v}.$$

Next, suppose further A is formed by a family of parallel hypersurfaces on the sphere. In general, given Γ^{n-1} as a hypersurface in $\mathbb{S}^n$, then $A = \{\bigcup_{0\le d\le T} \Gamma_d : \Gamma_d = \Gamma + d\nu_\Gamma\}$, where ν_Γ is the normal vector when we consider $\Gamma \subset \mathbb{S}^n$ and T small so that every point on the surface avoids its focal point.

Notice that d is exactly the distance between two hypersurfaces Σ_0 and Σ_d on the sphere. If we choose orthonormal frame $\{e_i\}_{i=1}^n$ so that $e_n = \nu_\Gamma$, then we have the following Lemma.

Lemma 1. *Let $\Gamma^{n-1} \subset \mathbb{S}^n$ with second fundamental form θ and d be the distance function from Γ on $\mathbb{S}^n$. Then*

$$\nabla_\alpha \nabla_\beta d = \begin{cases} 0 & \text{if } \alpha \text{ or } \beta = n \\ \theta_{\alpha\beta} & \text{otherwise} \end{cases}$$

Proof. *By $e_n = \nu_{\Sigma_0} = \partial_d$,*

$$\nabla_\alpha d = \begin{cases} 0 & \text{if } \alpha \neq n \\ 1 & \text{otherwise.} \end{cases}$$

Hence

$$\nabla_\alpha \nabla_\beta d = e_\alpha e_\beta d - \Gamma^\gamma_{\alpha\beta} \nabla_\gamma d = -\Gamma^n_{\alpha\beta}.$$

However,

$$\Gamma^n_{\alpha n} =< \partial_n, \nabla_\alpha \partial_n >= \frac{1}{2}\partial_\alpha < \partial_n, \partial_n >= 0,$$

and for $\alpha, \beta \neq n$

$$\Gamma^n_{\alpha\beta} =< e_n, \nabla_\alpha e_\beta >= -\theta_{\alpha\beta}.$$

Therefore, the Lemma follows.

In this light, if we assume the self-shrinker M on $A = \bigcup_{0\le d\le T}\Gamma_d$ to approach the cone containing Γ with the tip at the origin, then it is reasonable to consider the restriction $v(z) = v(d)$. As a consequence $\lim_{d\to 0} v(d) = +\infty$. Thus,

$$
\begin{aligned}
&(\sigma^{ij} - \frac{v^i v^j}{1+|\nabla v|^2})\nabla_i\nabla_j v \\
&= (\sigma^{ij} - \frac{v^i v^j}{1+|\nabla v|^2})(v''\nabla_i d\nabla_j d + v'\nabla_i\nabla_j d) \\
&= (1 - \frac{(v')^2|\nabla d|^2}{1+(v')^2})v'' + (\sigma^{ij} - \frac{v^i v^j}{1+|\nabla v|^2})\theta_{ij}v' \\
&= \frac{v''}{1+(v')^2} + H(d)v'.
\end{aligned}
$$

The self-shrinker equation is reduced to a second order ordinary differential equation:

$$
\frac{v''}{1+(v')^2} + H(d)v' = n - \frac{1}{2}e^{2v}.
$$

Then, by using change of variable $v(d) = \frac{-1}{2}\log g(d)$, we get

$$
\begin{aligned}
v'(d) &= \frac{-g'}{2g}, \\
v''(d) &= \frac{-[gg'' - (g')^2]}{2g^2}, \\
n - \frac{1}{2}e^{2v} &= n - \frac{1}{2g},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{v''}{1+(v')^2} + H(d)v' &= \frac{-[gg''-(g')^2]}{2g^2}\cdot\frac{1}{1+(\frac{-g'}{2g})^2} + H(d)\cdot(\frac{-g'}{2g}) \\
&= \frac{-[gg''-(g')^2]}{2g^2 + \frac{1}{2}(g')^2} - \frac{Hg'}{2g} \\
&= \frac{-2gg''}{4g^2+(g')^2} + \frac{2(g')^2}{4g^2+(g')^2} - \frac{Hg'}{2g}.
\end{aligned}
$$

Therefore, the equation transforms into

$$
\frac{-2gg''}{4g^2+(g')^2} + \frac{2(g')^2}{4g^2+(g')^2} - \frac{Hg'}{2g} = n - \frac{1}{2g},
$$

or equivalently,

$$\frac{1}{2g}[1-H(d)g'(d)]+2\frac{(g')^2-gg''}{(g')^2+4g^2}=n, \tag{2.3.1}$$

with initial condition

$$g(0)=\lim_{d\to 0}g(d)=\lim_{d\to 0}e^{-2v(d)}=0,$$

and it implies $g'(0)=\frac{1}{H(0)}>0$.

In order to simplify the analysis, we set $s=g(d)$ with inverse $d=\gamma(s)$. By the inverse function theorem, we calculate

$$g'(d)=\frac{1}{\gamma'(s)},g''(d)=\frac{-\gamma''(s)}{(\gamma'(s))^3}.$$

Hence equation (2.3.1) become

$$\frac{1}{2s}[1-H(\gamma)\frac{1}{\gamma'}]+2\frac{\frac{1}{(\gamma')^2}+s\frac{\gamma''}{(\gamma')^3}}{\frac{1}{(\gamma')^2}+4s^2}=n \tag{2.3.2}$$

$$\Rightarrow (1-2ns)\gamma'-H(\gamma)+4\frac{s\gamma'+s^2\gamma''}{1+4s^2(\gamma')^2}=0, \tag{2.3.3}$$

with initial condition $\gamma(0)=0,\gamma'(0)=H(0)>0$. Later on, we will seek for smooth solution of equation (2.3.3) on a uniform interval $[0,s_0]$ with $0<\gamma(s)<d_0$ and $\gamma'(s)>0$. However, we can determine $\gamma^{(k)}(0)$ by the equation so that we have formal power series solution

$$\sum_{k=1}^{\infty}\frac{A_k}{k!}s^k,$$

where $\{A_k\}$ are recursuvely define

$$A_{k+1}=(-4k^2+(2n+H'(0)k))A_k+P(A_1,\cdots,A_{k-1}),$$

and P is a polynomial.

2.4 Faà di Bruno Formula

The chain rule is well-known for its convenience to calculate the derivate for a composite function. In order to obtain higher derivatives, a similar formula had been developed and known as Faà di Bruno formula:

$$(\frac{d}{ds})^m f(g(s)) = \sum_{l=1}^{m} \sum_{b\in A_{m,l}} \frac{m!}{b_1!b_2!\cdots b_m!} f^{(l)}(g(s))\Pi_{k=1}^{m}(\frac{g^{(k)}(s)}{k!})^{b_k},$$

where $A_{m,l} = \{(b_1, b_2, \cdots, b_m) : \sum_{j=1}^{m} b_j = 1, \sum j = 1^m j \cdot b_j = m\} \subset \mathbb{N}^m$.

One important variation of Faà di Bruno formula we are going to use in this article is due to Yamanaka[32]:

$$\begin{aligned}(\frac{d}{ds})^m f(g(s)) &= \sum_{l=1}^{m} C_l^m f^{(l)}(g(s))\{(\frac{d}{ds})^{m-l}(\int_0^1 g'(s+h\theta))d\theta)^l\}|_{h=0} \\ &= f'(g(s))g^{(m)}(s) + \sum_{l=2}^{m} C_l^m f^{(l)}(g(s))\{(\frac{d}{dh})^{m-l}(\int_0^1 g'(s+h\theta))d\theta)^l\}|_{h=0}.\end{aligned}$$

For more information, the author recommands a survey paper by W.P. Johnson[19].

Suppse we choose $\eta(s) = \frac{1}{1+4s^2}$ and write $\eta^{(p)} = (-2)^p \cdot p! \cdot \Lambda_{p+1}(s)$, where

$$\Lambda_{p+1}(s) = \frac{\sin[(p+1)\arcsin(\frac{1}{\sqrt{1+4s^2}})]}{(1+4s^2)^{\frac{p+1}{2}}}.$$

Then we can compute the amount A_{k+1}^{ϵ}:

$$\begin{aligned}(\frac{d}{ds})^m \arctan(2s\gamma'(s)) &= \sum_{l=1}^{m} \sum_{b\in A_{m,l}} \frac{m!}{b_1!b_2!\cdots b_m!}(\frac{d}{dx})^l \arctan(2x)|_{x=s\gamma'}\Pi_{k=1}^{m}(\frac{(\frac{d}{ds})^k(s\gamma')}{k!})^{b_k} \\ &= \sum_{l=1}^{m} \sum_{b\in A_{m,l}} \frac{m!}{b_1!b_2!\cdots b_m!} 2(\frac{d}{dx})^{l-1}\eta(x)|_{x=s\gamma'}\Pi_{k=1}^{m}(\frac{(\frac{d}{ds})^k(s\gamma')}{k!})^{b_k} \\ &= \sum_{l=1}^{m} \sum_{b\in A_{m,l}} \frac{m!}{b_1!b_2!\cdots b_m!}(-1)^{l-1}2^l(l-1)!\Lambda_l(s\gamma')\Pi_{k=1}^{m}(\frac{s\gamma^{(k+1)}+k\gamma^{(k)}}{k!})^{b_k}.\end{aligned}$$

$$
\begin{aligned}
A^{\epsilon}_{k+1} &= (\frac{d}{ds})^k[H(\gamma) - 2k\arctan(2s\gamma')](0) \\
&= \sum_{l=1}^{k}\sum_{b\in A_{k,l}} H^{(l)}(0)\frac{k!}{b_1!b_2!\cdots b_k!}\Pi_{j=1}^{k}(\frac{\gamma^j(0)}{j!})^{b_j} \\
&- 2k\sum_{l=1}^{k}\sum_{b\in A_{k,l}} (-1)^{l-1}2^l(l-1)!\Lambda_l(0)\frac{k!}{b_1!b_2!\cdots b_k!}\Pi_{j=1}^{k}(\frac{\gamma^j(0)}{(j-1)!})^{b}_{j} \\
&= \sum_{l=1}^{k}\sum_{b\in A_{k,l}} (H^{(l)}(0) + (-1)^l 2^{l+1}k(l-1)!\Lambda_l(0)\Pi_{j=1}^{k}j^{b_j})\frac{k!}{b_1!b_2!\cdots b_k!}\Pi_{j=1}^{k}(\frac{\gamma^j(0)}{j!})^{b_j}
\end{aligned}
$$

Lemma 2.

$$
\begin{aligned}
(\frac{d}{ds})^m\eta(s\gamma'(s)) &= \sum_{l=1}^{m} C_l^m(-2)^l l!\Lambda_{l+1}(s\gamma')\{\frac{d}{dh})^{m-l}(\int_0^1 s\gamma''(s+h\theta))d\theta + \gamma'(s+h))^l\} \\
&= -8s\gamma'(s)\eta^2(s\gamma')(\frac{d}{ds})^m(s\gamma'(s)) \\
&+ \sum_{l=2}^{m} C_l^m(-2)^l l!\Lambda_{l+1}(s\gamma')\{\frac{d}{dh})^{m-l}(\int_0^1 s\gamma''(s+h\theta))d\theta + \gamma'(s+h))^l\}
\end{aligned}
$$

Proof. *By the aforementioned Yamanaka formula, choose $g(s) = s\gamma'(s)$ and use the integration by parts:*

$$
\begin{aligned}
\int_0^1 g'(s+h\theta)d\theta &= \int_0^1 ((s+h\theta)\gamma''(s+h\theta) + \gamma'(s+h\theta))d\theta \\
&= \int_0^1 (s+h\theta)\gamma''(s+h\theta)d\theta + \theta\gamma'(s+h\theta)|_{\theta=0} - \int_0^1 \theta[\frac{d}{d\theta}\gamma'(s+h\theta)]d\theta \\
&= \int_0^1 s\gamma''(s+h\theta)d\theta + \gamma'(s+h).
\end{aligned}
$$

Lemma 3. *Suppose $|(\frac{d}{ds})^j u(s)| \le M^{j-1}\frac{(j!)^2}{(j+1)^2}$ on $[0,s_0]$ for $0 \le j \le m$. Then*

$$
|(\frac{d}{ds})^j u(s)^l| \le (\frac{C}{M})^{l-1}M^{j-1}\frac{(j!)^2}{(j+1)^2}
$$

on $[0,s_0]$ for $0 \le j \le m$, where $l \in \mathbb{N}$ and $C = \frac{2\pi^2}{3}$.

Proof. *We will use induction on l and $l = 1$ is trivial. By hypothesis, $|(\frac{d}{ds})^j u(s)^k| \le (\frac{C}{M})^{k-1}M^{j-1}\frac{(j!)^2}{(j+1)^2}$*

is true for $k \leq l$ and $j \leq m$, therefore,

$$\begin{aligned}
|(\frac{d}{ds})^j u(s)^{l+1}| &= |\sum_{i=0}^{j} C_i^j [(\frac{d}{ds})^i u(s)^l] \cdot [(\frac{d}{ds})^{j-i} u(s)]| \\
&\leq \sum_{i=0}^{j} C_i^j (\frac{C}{M})^{l-1} M^{i-1} \frac{(i!)^2}{(i+1)^2} M^{j-i-1} \frac{[(j-i)!]^2}{(j-i+1)^2} \\
&= (\frac{C}{M})^{l-1} M^{j-2} \frac{j!}{(j+2)^2} \sum_{i=0}^{j} i!(j-i)!(\frac{1}{i+1} + \frac{1}{j-i+1})^2 \\
&\leq (\frac{C}{M})^{l} M^{j-1} \frac{(j!)^2}{C(j+1)^2} \sum_{i=0}^{\infty} \frac{4}{i^2} = (\frac{C}{M})^{l} M^{j-1} \frac{(j!)^2}{(j+1)^2}.
\end{aligned}$$

Lemma 4. *Assume that*

$$|(\frac{d}{ds})^{j+1}[s\gamma'(s)]| \leq M^{j-1} \frac{(j!)^2}{j+1}$$

on $[0, s_0]$ for $j \geq 1$ and define

$$u(h;s) = \int_0^1 s\gamma''(s+h\theta)d\theta + \gamma'(s+h), u_l(h;s) = (u(h;s))^l.$$

Then

$$|(\frac{d}{ds})^j u_l(h;s)|_{h=0}| \leq (\frac{C}{M})^{l-1} M^{j-1} \frac{(j!)^2}{(j+1)^2}.$$

***Proof*.** *Note that*

$$\begin{aligned}
(\frac{d}{ds})^j u(h;s) &= \int_0^1 s\theta^j \gamma^{(j+2)}(s+h\theta)d\theta + \gamma^{(j+1)}(s+h) \\
&= \int_0^1 (s+h\theta)\theta^j \gamma^{(j+2)}(s+h\theta)d\theta - \int_0^1 \theta^{j+1} \frac{d}{d\theta}[\gamma^{(j+1)}(s+h\theta)]d\theta + \gamma^{(j+!)}(s+h) \\
&= \int_0^1 \theta^j [(s+h\theta)\gamma^{(j+2)}(s+h\theta) + (j+1)\gamma^{(j+1)}(s+h\theta)]d\theta \\
&= \int_0^1 \theta^j (\frac{d}{ds})^{j+1}[s\gamma'(s)](s+h\theta)d\theta.
\end{aligned}$$

Therefore

$$|(\frac{d}{ds})^j u(h;s)| \leq \int_0^1 \theta^j M^{j-1} \frac{(j!)^2}{j+1} d\theta = M^{j-1} \frac{(j!)^2}{(j+1)^2},$$

we can apply Lemma 3 to get conclusion.

3

Self-shrinking Ends on Isoparametric Cones

3.1 ϵ-Regularized Problem

For the equation

$$(1-2ns)\gamma'(s) - H(\gamma) + 4\frac{s\gamma'(s) + s^2\gamma''(s)}{1+4s^2\gamma'^2(s)} = 0,$$
$$\gamma(0) = 0, s > 0,$$

we have a formal power series solution

$$\sum_{k=1}^{\infty} \frac{A_k}{k!} s^k,$$

where the coefficients A_k are obtained by a recursive formula. However, the equation is highly singular and degenerate. Therefore, we need to figure out a new strategy to deal with these difficulties.

One way to consider is to regularize the equation by adding a term $\epsilon\gamma''(s)$:

$$\epsilon\gamma''(s) + (1-2ns)\gamma'(s) - H(\gamma) + 4\frac{s\gamma'(s) + s^2\gamma''(s)}{1+4s^2\gamma'^2(s)} = 0,$$
$$\gamma(0) = 0, s > 0,$$

But then, we need a second initial condition in order to solve this ϵ-reularized problem. Naively, from the original equation, we know $\gamma'(0) = H(0)$. Unfortunately, if we add it as an initial condition, the regularized problem can't have a smooth solution, in fact, not even three times differentiable. As a result, for a fixed positive integer N, and $\epsilon > 0$ small enough, we consider the following regularized problem:

$$\epsilon\gamma''(s) + (1-2ns)\gamma'(s) - H(\gamma) + 4\frac{s\gamma'(s) + s^2\gamma''(s)}{1+4s^2\gamma'^2(s)} = 0, \tag{3.1.1}$$

$$\gamma(0) = 0, \gamma'(0) = B(\epsilon), s > 0, \tag{3.1.2}$$

where $B(\epsilon) = \sum_{i=0}^{N} B_i\epsilon^i$, $B_0 = H(0)$ and B_i's will be determined by the next theorem.

In this section, we first show that we could choose B_i recursively so that the regularized problem can approach the original problem well.

THEOREM 2. *There exists $B_1, B_2, \cdots, B_N$ such that*

$$\gamma^{(k)}(0) = A_k + ((-1)^{k-1}B_k + \phi_k(B_1, B_2, \cdots, B_N))\epsilon + O(\epsilon^2), k = 1, 2, \cdots, N+1.$$

Moreover, $|B_k|, |\gamma^{(k)}| \le C(H, N)$ independent of ϵ.

Proof. *According to (3.1.1), we differentiate it k-times and evaluate at $s = 0$:*

$$\begin{aligned}
(\frac{d}{ds})^k(1-2ns)\gamma'(0) &= \gamma^{(k+1)}(0) - 2nk\gamma^{(k)}(0), \\
(\frac{d}{ds})^k(4\frac{s^2\gamma'' + s\gamma'}{1+4s^2\gamma'^2})(0) &= 2(\frac{d}{ds})^k(s\frac{d}{ds})\arctan(2s\gamma')(0) \\
&= 2(s\frac{d}{ds} + k)(\frac{d}{ds})^k\arctan(2s\gamma')(0) \\
&= 2k(\frac{d}{ds})^k\arctan(2s\gamma')(0).
\end{aligned}$$

Therefore, we get a recursive formula

$$\epsilon\gamma^{(k+2)}(0) = -\gamma^{(k+1)}(0) + 2nk\gamma^{(k)}(0) + (\frac{d}{ds})^k(H(\gamma) - 2k\arctan(2s\gamma'))(0) \tag{3.1.3}$$

$$:= -\gamma^{(k+1)}(0) + A^{\epsilon}_{k+1}, \tag{3.1.4}$$

where $A^{\epsilon}_{k+1} = 2nk\gamma^{(k)}(0) + (\frac{d}{ds})^k(H(\gamma) - 2k\arctan(2s\gamma'))(0)$. Notice that, the last two terms only involve the differential of γ up to k-times and we will denote it by A^{ϵ}_{k+1}. A more explicit expression of the right hand side was obtained by Faà di Bruno formula. Note that the coefficients A_k of the formal solution can be derived from setting $\epsilon = 0$ and $A_1 = \gamma'(0) = H(0)$. This leads to the simple *but important observation:*

$$A_{k+1} = F(A_1, \cdots, A_k) = A^{\epsilon}_{k+1}|_{\epsilon=0}. \tag{3.1.5}$$

Also, from this formula, if we apply $k = 0$, $\gamma''(0) + \gamma'(0) - H(0) = 0$, *then we get* $\gamma''(0) = -\sum_{i=1}^{N} B_i \epsilon^{i-1}$. *Thus, in order to approximate the original problem well, we choose* $B_1 = -A_2$. *Inductively, we can determine* $B_0, B_1, \cdots, B_k$ *so that*

$$\gamma^{(l)} = A_l + [(-1)^{l-1} B_l + \phi_l(B_1, \cdots, B_{l-1})]\epsilon + O(\epsilon^2), l = 2, \cdots, k+1.$$

Claim 1. $\gamma^{(k+2)}(0) = (-1)^{k+1} B_{k+1} + \phi(B_1, \cdots, B_k) + O(\epsilon)$.

To prove the claim, we will need a more explicit expression for A^{ϵ}_{k+1} *so that we can expand* $\epsilon\gamma^{(k+2)}$ *in terms of power of* ϵ.

By (3.1.4) and (3.1.5) and our induction hypothesis, the constant term in the expansion of $\epsilon\gamma^{(k+2)}$ *in power of* ϵ *is automatic* 0 *and the* ϵ *term begins with* $(-1)^{k+1} B_k + \phi_{k+1}(B1, \cdots, B_{k-1})$ *coming from* $\gamma^{k+1}(0)$. *It remains to verify that the* ϵ *terms of* A^{ϵ}_{k+1} *depends only on* $B_1, \cdots, B_{k-1}$. *But this is now obvious from Faa di Bruno formula we derived before and our induction hypothesis. Therefore,*

$$\epsilon\gamma^{(k+2)}(0) = (-A_{k+1} + A^{\epsilon}_{k+1}|_{\epsilon=0}) + [(-1)^{k+1} B_k + \phi(B_1, \cdots, B_{k-1})]\epsilon + O(\epsilon^2)$$

implies

$$\gamma^{(k+2)}(0) = (-1)^{k+1} B_k + \phi(B_1, \cdots, B_{k-1}) + O(\epsilon),$$

completing the proof.

3.2 Apriori Estimate

Now, we have control of the first $N+1$ derivatives of γ at the origin, we will inductively derive energy estimates that imply $0 < \gamma < d_0$, $\gamma'(s) > 0$, $|\gamma^{(k)}(s)| \le C_k$ for $k = 1, 2, \cdots, N$ independent of ϵ on a uniform interval $(0, s_0)$. We assume that ϵ small enough that

$$\frac{1}{2} H(0) \le \gamma'(0) \le 2H(0), |\gamma^{(k)}(0) - A_k| \le 1, \epsilon|\gamma^{(k)}| \le 1, k = 1, 2, \cdots, N+1.$$

In the following discussion, for convenience, C will denote a constant depending on H and dimension n which may change from line to line. We multiply the ϵ-regularized equation

$$\epsilon\gamma''(s) + (1-2ns)\gamma'(s) - H(\gamma) + 4\frac{s\gamma'(s) + s^2\gamma''(s)}{1+4s^2\gamma'^2(s)} = 0, \tag{3.2.1}$$

by γ' and integrate from 0 to s to obtain

$$\frac{\epsilon}{2}[(\gamma')^2(s) - (\gamma')^2(0)] + \frac{1}{2}\log\left[1+4s^2(\gamma')^2(s)\right] + \int_0^s (1-2nt)(\gamma')^2(t)dt = \bar{H}(\gamma), \tag{3.2.2}$$

where $\bar{H}'(t) = H(t)$ and $\bar{H}(0) = 0$. We restrict $0 < s < \frac{1}{4n}$. Since $\bar{H}(\gamma) \leq C\gamma$, by the Cauchy Schwarz inequality we obtain

$$[\gamma(s)]^2 = (\int_0^s \gamma'(t)dt)^2 \leq s\int_0^s (\gamma')^2 dt \leq s[\epsilon(\gamma')^2(0) + 2C\gamma(s)].$$

Hence $\gamma(s) \leq C(s + \sqrt{\epsilon s})$. Thus, by (3.2.1) and (3.2.2):

Lemma 1.

$$\begin{aligned}
&\gamma(s) \leq C(s+\sqrt{\epsilon s}),\\
&0 \leq s\gamma'(s) \leq C(s+\sqrt{\epsilon s})^{\frac{1}{2}},\\
&s^2|\gamma''(s)| \leq \gamma'(s) + C,\\
&\int_0^s (\gamma'(t))^2 dt \leq C.
\end{aligned}$$

3.2.1 First Derivative of the Equation

Next, we take the derivative of (3.2.1) and apply a similar argument, multiply by $\gamma''(s)$ and integrate the equation:

$$\begin{aligned}
&\frac{\epsilon}{2}[(\gamma''(s))^2 - (\gamma''(0))^2] + 2s^2\eta(s\gamma')[\gamma''(s)]^2\\
&+\int_0^s [(1-2nt) + 8t\eta(t\gamma') - 16t^3\eta(t\gamma')\gamma'](\gamma''(t))^2 dt\\
&+\int_0^s [4\eta(t\gamma') - 2n - H'(\gamma) - 32t^2\eta(t\gamma')\gamma']\gamma'\gamma'' dt = 0,
\end{aligned}$$

where $\eta(x) = \frac{1}{1+4x^2}$. By above lemma, it's clear that the third term can be estimated

$$\int_0^s [(1-2nt) + 8t\eta(t\gamma') - 16t^3\eta(t\gamma')\gamma'](\gamma''(t))^2 dt \geq \int_0^s (1-o(1))(\gamma''(t))^2 dt.$$

Therefore, there exists $s_0 > 0$ uniformly that for $0 \leq s \leq s_0$,

$$\frac{\epsilon}{2}(\gamma''(s))^2 + \frac{2s^2[\gamma''(s)]^2}{1+4s^2[\gamma'(s)]^2} + \int_0^s (\gamma'')^2 dt \leq C,$$

and improves the lemma

$$|\gamma'(s) - \gamma'(0)| \leq C\sqrt{s}, s|\gamma''(s)| \leq C,$$
$$s^2|\gamma''(s)| \leq s^2|\gamma''(0)| + C\sqrt{s}.$$

3.2.2 Higher Derivative of the Equation

For the equation

$$\epsilon\gamma'' + 4\frac{s^2\gamma'' + s\gamma'}{1+4s^2(\gamma')^2} + (\widetilde{1} - 2ns)\gamma' - H(\gamma) = 0,$$

where we use $(\widetilde{1} - 2ns)$ to substitute $(1 - 2ns)$ for later use, take the differential k-times and use $\eta(x) = \frac{1}{1+4x^2}$:

$$\begin{aligned}
&\epsilon\gamma^{(k+2)} + 4(\frac{d}{ds})^k[(s^2\gamma'' + s\gamma')\eta(s\gamma')] + (\widetilde{1} - 2ns)\gamma^{(k+1)} - 2nk\gamma^{(k)} - (\frac{d}{ds})^k H(\gamma) \\
=&\epsilon\gamma^{(k+2)} + 4\sum_{l=0}^{k} C_l^k(\frac{d}{ds})^l(s^2\gamma'' + s\gamma')(\frac{d}{ds})^{k-l}\eta(s\gamma') + (\widetilde{1} - 2ns)\gamma^{(k+1)} - 2nk\gamma^{(k)} - (\frac{d}{ds})^k H(\gamma) \\
=&\epsilon\gamma^{(k+2)} + 4\sum_{l=1}^{k-2} C_l^k(s^2\gamma^{(l+2)} + (2l+1)s\gamma^{(l+1)} + l^2\gamma^{(l)})(\frac{d}{ds})^{k-l}\eta(s\gamma') \\
&+ 4k(s^2\gamma^{(k+1)} + (2k-1)s\gamma^{(k)} + (k-1)^2\gamma^{(k-1)})\frac{d}{ds}\eta(s\gamma') \\
&+ 4(s^2\gamma^{(k+2)} + (2k+1)s\gamma^{(k+1)} + k^2\gamma^{(k)})(\eta(s\gamma') \\
&+ 4(s^2\gamma'' + s\gamma')(\frac{d}{ds})^k\eta(s\gamma') + (\widetilde{1} - 2ns)\gamma^{(k+1)} - 2nk\gamma^{(k)} - (\frac{d}{ds})^k H(\gamma) = 0.
\end{aligned}$$

Next, multiple by $\gamma^{(k+1)}$ and take integral on the interval $[0, s]$:

$$
\begin{aligned}
&\epsilon \int_0^s \gamma^{(k+1)}\gamma^{(k+2)}dt + 4\sum_{l=1}^{k-2} C_l^k \int_0^s \gamma^{(k+1)}(t^2\gamma^{(l+2)} + (2l+1)t\gamma^{(l+1)} + l^2\gamma^{(l)})(\frac{d}{dt})^{k-l}\eta(t\gamma')dt \\
&+ 4k\int_0^s \gamma^{(k+1)}(t^2\gamma^{(k+1)} + (2k-1)t\gamma^{(k)} + (k-1)^2\gamma^{(k-1)})\frac{d}{dt}\eta(t\gamma')dt \\
&+ 4\int_0^s \gamma^{(k+1)}(t^2\gamma^{(k+2)} + (2k+1)t\gamma^{(k+1)} + k^2\gamma^{(k)})\eta(t\gamma')dt \\
&+ 4\int_0^s \gamma^{(k+1)}(t^2\gamma'' + t\gamma')(\frac{d}{dt})^k\eta(t\gamma')dt + \int_0^s \gamma^{(k+1)}(\widetilde{1} - 2nt)\gamma^{(k+1)}dt \\
&- 2nk\int_0^s \gamma^{(k+1)}\gamma^{(k)}dt - \int_0^s \gamma^{(k+1)}(\frac{d}{dt})^k H(\gamma)dt = 0
\end{aligned}
$$

Simplify the result by collecting similar terms and add an auxilirary term:

$$
\begin{aligned}
&\frac{\epsilon}{2}[(\gamma^{(k+1)})^2(s)-(\gamma^{(k+1)})^2(0)]\\
&+4\sum_{l=1}^{k-2}C_l^k\int_0^s\gamma^{(k+1)}(t^2\gamma^{(l+2)}+(2l+1)t\gamma^{(l+1)}+l^2\gamma^{(l)})(\frac{d}{dt})^{k-l}\eta(t\gamma')dt\\
&+4k\int_0^s\gamma^{(k+1)}(t^2\gamma^{(k+1)}+(2k-1)t\gamma^{(k)}+(k-1)^2\gamma^{(k-1)})\frac{d}{dt}\eta(t\gamma')dt\\
&+2\int_0^s t^2\eta(t\gamma')\frac{d}{dt}[\gamma^{(k+1)}]^2dt+4\int_0^s\gamma^{(k+1)}((2k+1)t\gamma^{(k+1)}+k^2\gamma^{(k)})\eta(t\gamma')dt\\
&+4\int_0^s\gamma^{(k+1)}(t^2\gamma''+t\gamma')(\frac{d}{dt})^k\eta(t\gamma')dt+\int_0^s\gamma^{(k+1)}(\widetilde{1}-2nt)\gamma^{(k+1)}dt\\
&-2nk\int_0^s\gamma^{(k+1)}\gamma^{(k)}dt-\int_0^s\gamma^{(k+1)}(\frac{d}{dt})^kH(\gamma)dt\\
&+32\int_0^s\gamma^{(k+1)}(t^2\gamma''+t\gamma')t\gamma'\eta^2(t\gamma')(t\gamma^{(k+1)}+k\gamma^{(k)})dt\\
&-32\int_0^s\gamma^{(k+1)}(t^2\gamma''+t\gamma')t^2\gamma'\eta^2(t\gamma')\gamma^{(k+1)}dt-32k\int_0^s\gamma^{(k+1)}(t^2\gamma''+t\gamma')t\gamma'\eta^2(t\gamma')\gamma^{(k)}dt\\
=&\frac{\epsilon}{2}[(\gamma^{(k+1)})^2(s)-(\gamma^{(k+1)})^2(0)]+2s^2\eta(s\gamma')(\gamma^{(k+1)})^2\\
&+4\sum_{l=1}^{k-2}C_l^k\int_0^s\gamma^{(l+1)}(t^2\gamma^{(l+2)}+(2l+1)t\gamma^{(l+1)}+l^2\gamma^{(l)})(\frac{d}{dt})^{k-l}\eta(t\gamma')dt\\
&+\int_0^s\{-2\frac{d}{dt}(t^2\eta(t\gamma'))+8kt\eta(t\gamma')[1-4t^2\eta(t\gamma')][(\gamma')^2+t\gamma'\gamma'']\}(\gamma^{(k+1)})^2dt\\
&+4\int_0^s\gamma^{(k+1)}(t^2\gamma''+t\gamma')\{(\frac{d}{dt})^k\eta(t\gamma')+8t\gamma'\eta^2(t\gamma')[t\gamma^{(k+1)}+k\gamma^{(k)}]\}dt\\
&-32\int_0^s(t^2\gamma''+t\gamma')t^2\gamma'\eta^2(t\gamma')[\gamma^{(k+1)}]^2dt-32k\int_0^s(t^2\gamma''+t\gamma')t\gamma'\eta^2(t\gamma')\gamma^{(k)}\gamma^{(k+1)}dt\\
&+\int_0^s[4k^2\eta(t\gamma')+4k(2k-1)t\frac{d}{dt}\eta(t\gamma')-2nk]\gamma^{(k)}\gamma^{(k+1)}dt\\
&+4k(k-1)^2\int_0^s\frac{d}{dt}\eta(t\gamma)\gamma^{(k-1)}\gamma^{(k+1)}dt+\int_0^s(\widetilde{1}-2nt)[\gamma^{(k+1)}]^2dt\\
&-\int_0^s\gamma^{(k+1)}(\frac{d}{dt})^kH(\gamma)dt=0.
\end{aligned}
$$

For later use, we will denote the following terms:

$$
\begin{aligned}
I_0(\epsilon,\widetilde{1}) =& \int_0^s (\widetilde{1}-2nt)[\gamma^{(k+1)}]^2 dt - 32\int_0^s (t^2\gamma''+t\gamma')t^2\gamma'\eta^2(t\gamma')[\gamma^{(k+1)}]^2 dt \\
&+ \int_0^s \{-2\frac{d}{dt}(t^2\eta(t\gamma')) + 8kt\eta(t\gamma')[1-4t^2\eta(t\gamma')][(\gamma')^2+t\gamma'\gamma'']\}(\gamma^{(k+1)})^2 dt \\
I_1(\epsilon) =& \int_0^s \gamma^{(k+1)}(\frac{d}{dt})^k H(\gamma)dt \\
I_2(\epsilon) =& 4\sum_{l=1}^{k-2} C_l^k \int_0^s \gamma^{(l+1)}(t^2\gamma^{(l+2)}+(2l+1)t\gamma^{(l+1)}+l^2\gamma^{(l)})(\frac{d}{dt})^{k-l}\eta(t\gamma')dt \\
I_3(\epsilon) =& 4\int_0^s \gamma^{(k+1)}(t^2\gamma''+t\gamma')\{(\frac{d}{dt})^k\eta(t\gamma')+8t\gamma'\eta^2(t\gamma')[t\gamma^{(k+1)}+k\gamma^{(k)}]\}dt \\
I_4(\epsilon) =& 4k(k-1)^2\int_0^s \frac{d}{dt}\eta(t\gamma)\gamma^{(k-1)}\gamma^{(k+1)}dt - 32k\int_0^s (t^2\gamma''+t\gamma')t\gamma'\eta^2(t\gamma')\gamma^{(k)}\gamma^{(k+1)}dt \\
&+ \int_0^s [4k^2\eta(t\gamma')+4k(2k-1)t\frac{d}{dt}\eta(t\gamma')-2nk]\gamma^{(k)}\gamma^{(k+1)}dt
\end{aligned}
$$

Therefore, we have

$$
I_0(\epsilon,1) = \int_0^s (1+o(1))[\gamma^{(k+1)}(t)]dt. \tag{3.2.3}
$$

3.2.3 Main Estimates

THEOREM 3. *Let $\gamma=\gamma(s;\epsilon,N)$ be a solution of the ϵ-regularized initial value problem*

$$
\begin{aligned}
&\epsilon\gamma''(s)+(1-2ns)\gamma'(s)-H(\gamma)+4\frac{s\gamma'(s)+s^2\gamma''(s)}{1+4s^2\gamma'^2(s)}=0, \\
&\gamma(0)=0, \gamma'(0)=B(\epsilon), s>0,
\end{aligned}
$$

on the interval $[0,s]$. There exists s_0 sufficiently small, independent of ϵ and N so that

$$
\|\gamma^{(j)}\|_{L^2[0,s]} + |(\frac{d}{ds})^{j-1}(s\gamma'(s))| \le C(N), 1\le j\le N+1, s\in[0,s_0]. \tag{3.2.4}
$$

Proof. *We will prove (3.2.4) by induction. We have already proved the estimate for $j=1,2$, therefore, $s\gamma''+\gamma'$ and $\frac{d}{ds}\eta(s\gamma')$ are bounded. Moreover, the relation (3.2.3) holds. In the following discussion $C(N)$ will be a constant independent of ϵ and s_0 which may change from line to line. Suppose (3.2.4) holds for $j=1,2,\cdots,k$. Note that*

$$
s^2\gamma^{(l+2)}+(2l+1)s\gamma^{(l+1)}+l^2\gamma^{(l)} = s(\frac{d}{ds})^{l+1}(s\gamma')+l(\frac{d}{ds})^l(s\gamma')
$$

is uniformly bounded for $1 \le l \le k-2$ *and so is* $(\frac{d}{ds})^{k-l}\eta(s\gamma')$ *by the Faà di Bruno formula. Also, we have*

$$(\frac{d}{ds})^k\eta(s\gamma') = -8s\gamma'\eta^2(s\gamma')[s\gamma^{(k+1)} + k\gamma^{(k)}] + \text{ bounded terms.}$$

Hence the term $I_3(\epsilon)$

$$\begin{aligned}&4\int_0^s \gamma^{(k+1)}(t^2\gamma'' + t\gamma')\{(\frac{d}{ds})^k\eta(t\gamma') + 8t\gamma'\eta^2(t\gamma')[t\gamma^{(k+1)} + k\gamma^{(k)}]\}dt \\ &= O(C(N)\int_0^s |\gamma^{(k+1)}|dt) = O(\frac{1}{10}\int_0^s (\gamma^{(k+1)})^2 dt + C(N)).\end{aligned}$$

Moreover, $(\frac{d}{ds})^k H(\gamma) = H'(\gamma)\gamma^{(k)} + O(C(N))$*,* I_1, I_2, I_4 *can be handled in the same manner. Thus, for* s_0 *sufficiently small, we derive from*

$$\begin{aligned}&\frac{\epsilon}{2}[(\gamma^{(k+1)})^2(s) - (\gamma^{(k+1)})^2(0)] + 2s^2\eta(s\gamma')(\gamma^{(k+1)})^2 \\ &\quad + I_0 + I_1 + I_2 + I_3 + I_4 = 0, \\ &\frac{\epsilon}{2}(\gamma^{(k+1)})^2 \ge 0, s\gamma' \le C(s + \sqrt{\epsilon s})^{\frac{1}{2}},\end{aligned}$$

and conclude that

$$\begin{aligned}&(\|\gamma^{(j)}\|_{L^2[0,s]} + |(\frac{d}{ds})^{j-1}(s\gamma'(s))|)^2 \\ &\le 2\{\int_0^s (\gamma^{(k+1)})^2 dt + 2\eta(s\gamma')[s\gamma^{(k+1)}]^2\} \le C(N).\end{aligned}$$

3.3 Uniqueness and Existence

In this section, we prove the uniqueness and existence of the ϵ-regularized problem and the origin problem by using the results of previous sections. As a consequence, we will also show the existence of self-shrinking end over an isoparametric cone by assuming the solution is of second Gevery Class which we will discuss in later sections.

THEOREM 4. *Assume* $H \in C^1[0, d_0)$ *and* $H'(d) < 0$*, where* d_0 *is where* Γ_{d_0} *is the unique minimal surface in the family of parallel hypersurfaces. Then there is at most one* $C^2[0, s_0)$ *solution to the*

ϵ-regularized problem

$$\begin{cases} \epsilon\gamma''(s) + (1-2ns)\gamma'(s) - H(\gamma) + 4\frac{s\gamma'(s)+s^2\gamma''(s)}{1+4s^2\gamma'^2(s)} = 0, s > 0, \\ \gamma(0) = 0, \gamma'(0) = B(\epsilon) \end{cases},$$

and the original problem

$$\begin{cases} (1-2ns)\gamma'(s) - H(\gamma) + 4\frac{s\gamma'(s)+s^2\gamma''(s)}{1+4s^2\gamma'^2(s)} = 0, s > 0, \\ \gamma(0) = 0 \end{cases}.$$

Proof. *Suppose there are two solutions γ_1, γ_2, and let $u(s) = \gamma_1(s) - \gamma_2(s)$. As a result, $u(0) = u'(0) = 0$. By the equation, we obtain*

$$\epsilon u''(s) + (1-2ns)u'(s) + 2s\frac{d}{ds}[\arctan(2s\gamma_1') - \arctan(2s\gamma_2')] = H(\gamma_1) - H(\gamma_2).$$

Next, using mean value theorem,

$$\begin{aligned} &2s\frac{d}{ds}[\arctan(2s\gamma_1') - \arctan(2s\gamma_2')] = 4s\frac{d}{ds}[2sa(s)u'(s)], \\ &H(\gamma_1) - H(\gamma_2) = b(s)u(s), \end{aligned}$$

where $a(s) = \int_0^1 \frac{d\theta}{1+4s^2[\gamma_\theta']^2}$, $b(s) = \int_0^1 H(\gamma_\theta)d\theta$, and $\gamma_\theta = \gamma_2 + \theta u$, we have

$$\begin{aligned} &\epsilon u''(s) + (1-2ns)u'(s) + 4s\frac{d}{ds}[2sa(s)u'(s)] = b(s)u(s), \\ &\Rightarrow \epsilon u''(s)u'(s) + (1-2ns)[u'(s)]^2 + 4su'(s)\frac{d}{ds}[2sa(s)u'(s)] = b(s)u(s)u'(s). \\ &\Rightarrow \frac{\epsilon}{2}[\gamma'(s)]^2 + \int_0^s (1-2nt+4t^2a'(t))[u'(t)]^2dt + 4a(s)s^2[u'(s)]^2 = \int_0^s b(s)u(t)u'(t)dt. \end{aligned}$$

Now, by $a'(s) = O(s)$, $1-2nt+4t^2a'(t) > 0$ and $a(s) > \frac{1}{2}$ for $0 < t < s$ small enough. Moreover, by Sobolev inequality and $b(s)$ is bounded

$$|\int_0^s b(t)u(t)u'(t)dt| \le C[\int_0^s u^2(t)dt]^{\frac{1}{2}}[\int_0^s (u'(t))^2dt]^{\frac{1}{2}} \le Cs\int_0^s [u'(t)]^2dt.$$

However, combining these two facts, the equation implies

$$\int_0^s [u'(t)]^2dt \le Cs\int_0^s [u'(t)]^2dt.$$

Therefore, $u(s) = u(0) = 0$ *is required.*

THEOREM 5. *There is* s_0 *depending only on* H *such that the initial problem*

$$\begin{cases} (1-2ns)\gamma'(s) - H(\gamma) + 4\frac{s\gamma'(s)+s^2\gamma''(s)}{1+4s^2\gamma'^2(s)} = 0, s > 0, \\ \gamma(0) = 0 \end{cases}$$

has a unique solution $\gamma \in C^\infty[0, s_0]$*. Moreover,* $\gamma^{(k)}(0) = A_k$ *for all* k.

Proof. *According to standard existence theorem, for any integer* N*, there us an analytic solution* $\gamma_N(s;\epsilon)$ *of the* ϵ*-regularized initial problem on a uniform interval* $[0, s_0)$ *independent of* ϵ *and* N*. Moreover,*

$$\|\gamma_N\|_{C^{N+1}[0,s_0]} \leq C(N), \text{ and } \lim_{\epsilon\to 0} \gamma_N^{(k)}(0;\epsilon) = A_k.$$

Hence by taking limits and using uniqueness, there is a solution $\gamma(s)$ *with* $\gamma^{(k)}(0) = A_k$ *for* $k = 1, 2, \cdots, N$*. Since* N *is arbitrary, the theorem follows.*

Proof (proof of the main theorem)**.** *According to uniqueness and the calculations of previous section, there exists a unique solution* $v(d) = \frac{-1}{2}\log g(d)$ *in a small interval* $(0, d_1)$ *to the equation*

$$\frac{v''}{1+v'^2} + v'H(d) = n - \frac{1}{2}e^{2v}. \tag{3.3.1}$$

That is the radial graph $\Sigma = \{e^{v(d(z))} : z \in A\}$*, where* $d(z)$ *is the distance function to* Γ *in* $A = \{z : 0 < d(z) \leq d_1 + \epsilon\}$ *is an end of a self-shrinker to the mean curvature flow. Moreover, assuming* $g(d) = e^{-2v(d)}$ *is in Gevrey Class* G^2*, since* $g(d)$ *is the inverse function of* $\gamma(s)$*. It remains to show* $v(d)$ *exists on the interval* $(0, d_0 + \epsilon)$ *for some small* $\epsilon > 0$ *where the parallel hypersurface to* Γ*,* $d(z) = d_0$*, is the unique minimal hypersurface of the family. To see this, we multiply (3.3.1) by* $2v'(d)$ *and integrate from* $\frac{d_1}{2}$ *to* d *to obtain*

$$\log[1+(v')^2] + 2\int_{\frac{d_1}{2}}^{d} H(t)(v')^2 dt = 2nv(d) - \frac{1}{2}e^{2v} + C(d_1).$$

Note that the right hand side tends to negative infinity as $v \to \pm\infty$ *while the left hand side remains strictly positive as long as* $H \geq 0$*. It's know easily that the finction continues past* $d = d_0$*, completing the proof.*

3.4 Gevrey Class

For a function $f:\mathbb{R}\to\mathbb{R}$, we say it is of **Second Gevrey Class** $G^2[0,s_0]$ if

$$|f^{(k)}|\leq CM^k(k!)^2,$$

for some universal constants $C, M\geq 0$.

In this section, we are trying to show the unique solution $\gamma(s)$ we obtained last section of

$$\begin{cases} (1-2ns)\gamma'(s)-H(\gamma)+4\frac{s\gamma'(s)+s^2\gamma''(s)}{1+4s^2\gamma'(s)^2}=0, s>0 \\ \gamma(0)=0 \end{cases} \tag{3.4.1}$$

is in the second Gevrey class $G^2[0,s_0]$ for some constant s_0.

The first thing we notice here, if we scale the solution by λ, $\widetilde{\gamma}(s)=\gamma(\lambda s)$, then for $0\leq s\leq\frac{s_0}{\lambda}$, it satisfies

$$\begin{cases} (\frac{1}{\lambda}-2ns)\widetilde{\gamma}'(s)-H(\widetilde{\gamma})+4\frac{s\widetilde{\gamma}'(s)+s^2\widetilde{\gamma}''(s)}{1+4s^2\widetilde{\gamma}'(s)^2}=0 \\ \widetilde{\gamma}(0)=0 \end{cases},$$

$$\frac{d}{ds}(s\widetilde{\gamma})=\widetilde{\gamma}'(s)+s\widetilde{\gamma}''(s)=\lambda\gamma'(\lambda s)+\lambda^2 s\gamma''(\lambda s)=\lambda\frac{d}{dt}(t\gamma'(t))|_{t=\lambda s}, and$$

$$\|\widetilde{\gamma}''\|_{L^2[0,\frac{s_0}{\lambda}]}=(\int_{[0,\frac{s_0}{\lambda}]}\lambda^4\gamma''(\lambda s)ds)^{\frac{1}{2}}=(\int_{[0,s_0]}\lambda^4\gamma''(t)\frac{dt}{\lambda})^{\frac{1}{2}}=\lambda^{\frac{3}{2}}\|\gamma''\|_{[0,s_0]}.$$

Recall that

$$\begin{aligned} I_0(\epsilon,\widetilde{1})=&\int_0^s(\widetilde{1}-2nt)[\gamma^{(k+1)}]^2dt-32\int_0^s(t^2\gamma''+t\gamma')t^2\gamma'\eta^2(t\gamma')[\gamma^{(k+1)}]^2dt \\ &+\int_0^s\{-2\frac{d}{dt}(t^2\eta(t\gamma'))+8kt\eta(t\gamma')[1-4t^2\eta(t\gamma')][(\gamma')^2+t\gamma'\gamma'']\}(\gamma^{(k+1)})^2dt, \end{aligned}$$

and we now use $\frac{1}{\lambda}$ to replace $\widetilde{1}$ and take limit on ϵ , then it's easy to see

$$0<I_0(1)=\lim_{\epsilon\to 0}I_0(\epsilon,1)\leq\lim_{\epsilon\to 0}I_(\epsilon,\frac{1}{\lambda})=I_0(\frac{1}{\lambda}),$$

for $s_0>0$ and $\lambda>0$ small enough, which is better when estimating the derivatives. Therefore, in this section, we will assume $\|\gamma''\|_{L^2[0,s_0]}+\frac{d}{ds}(s\gamma)$ is small.

THEOREM 6. *There exists $s_0>0$ small and $M>0$ large depending only on H such that if γ is the* *unique solution of ([3.4.1]), then*

$$\|\gamma^{(j)}\|_{L^2[0,s]} + |(\frac{d}{ds}^{j-1}[s\gamma'(s)])| \le M^{j-3}\frac{(j-2)!^2}{j-1} \tag{3.4.2}$$

on $[0, s_0]$ *for all* $j \ge 2$.

Now, the strategy of proving the above thoerem is by taking the limit on ϵ of the identity we obtained from previous section and use induction on j:

$$I_0 + 2s^2\eta(s\gamma')[\gamma^{(k+1)}]^2 + I_1 + I_2 + I_3 + I_4 = 0.$$

Since $I_0 = \int_0^s [1 + o(1)](\gamma^{k+1})^2 dt$, to estimate (3.4.2), it's equivalent to estimate I_1, I_2, I_3, I_4. In the following, we assume (3.4.2) is true for $j = 2, \cdots, k$.

PROPOSITION 1. *Assume the induction hypothesis, we have*

1. *For* $1 \le n \le k-1$,

$$|(\frac{d}{ds})^n(\eta(s\gamma'))| \le 16M^{n-2}\frac{(n-1)!^2}{n}, \tag{3.4.3}$$

2. *For some* $C > 0$,

$$|(\frac{d}{ds})^k\eta(s\gamma') + 8s\gamma'\eta^2(s\gamma')[\gamma^{(k+1)+k\gamma^{(k)}}]| \le CM^{k-4}\frac{k!(k-2)!}{(k-1)^2} \tag{3.4.4}$$

Proof. *From Faà di Bruno formula, induction hypothesis:*

$$\begin{aligned}
|(\frac{d}{ds})^n\eta(s\gamma'(s))| &= |\sum_{l=1}^{n} C_l^n(-2)^l l!\Lambda_{l+1}(s\gamma')\{\frac{d}{dh})^{n-l}(\int_0^1 s\gamma''(s+h\theta))d\theta + \gamma'(s+h))^l\}| \\
&\le \sum_{l=1}^{n} C_l^n 2^l l!|\Lambda_{l+1}(s\gamma')|(\frac{C}{M})^{l-1}M^{n-l-1}\frac{(n-l)!^2}{(n-l+1)^2} \\
&\le \frac{n!M^n}{C}\sum_{l=1}^{n}(\frac{2C}{M^2})^l\frac{(n-l)!}{(n-l+1)^2}
\end{aligned}$$

Next, choose M *large enough such that* $\frac{2C}{M^2} < 1$ *and* $\sum_{l=1}^{n}\frac{(n-l)!}{(n-l+1)^2} \le \frac{8(n-1)!}{n^2}$:

$$|(\frac{d}{ds})^n\eta(s\gamma'(s))| \le \frac{n!M^n}{C}\sum_{l=1}^{n}(\frac{2C}{M^2})\frac{(n-l)!}{(n-l+1)^2} \le 16M^{n-2}\frac{(n-1)!^2}{n}.$$

The proof of second part is similar.

The next four lemmas estimate I_1, I_2, I_3, I_4 under the induction hypothesis.

LEMMA 2. *Assume the induction hypothesis for $j = 2, 3, \cdots, k$, we have*

1. *For some constant C_1 depending on H,*

$$\|(\frac{d}{ds}H(\gamma))\|_{L^2[0,s]} \le C_1 M^{k-3}\frac{(k-2)!^2}{k-1} + \sqrt{s_0}C_1^2 C M^{k-4}(k-2)!^2.$$

2. *For some constant C_2 depending on H,*

$$|I_1| = |\int_0^s [(\frac{d}{dt})^k H(\gamma)]\gamma^{(k+1)}dt| \le C_2 M^{k-3}(k-2)!^2\|\gamma^{(k+1)}\|_{L^2[0,s]}.$$

Proof. *The first part implies the second part by using Schwarz inequality. Hence, we will focus on proving the first part.*

By Faà di Bruno formula,

$$(\frac{d}{ds})^k H(\gamma) = H'(\gamma)\gamma^{(k)} + \sum_{l=2}^{k} C_l^k H^{(l)}(\gamma)\{(\frac{d}{dh})^{k-l}[\int_0^1 \gamma'(s+h\theta)d\theta]^l\}|_{h=0}.$$

The first term is estimated by induction hypothesis,

$$\|H'(\gamma)\gamma^{(k)}\|_{L^2[0,s]} \le \|H'\|_{L^2[0,s]}\cdot\|\gamma^{(k)}\|_{L^2[0,s]} \le C_1 M^{k-3}\frac{(k-2)!^2}{k-1}.$$

Next, in order to estimate the second term, for $0 \le j \le k-2$, choose M large and use induction hypothesis:

$$\begin{aligned}
&(\frac{d}{dh})^j\int_0^1 \gamma'(s+h\theta)d\theta = \int_0^1 \theta^j\gamma^{(j+1)}(s+h\theta)d\theta \\
&\le (\int_0^1 \theta^{2j}d\theta)^{\frac{1}{2}}(\int_0^1 [\gamma^{(j+1)}(s+h\theta)]^2)^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{2j+1}}\|\gamma^{(j+1)}\|_{L^2[0,s_0-h]} \\
&\le \frac{M^{j-2}(j-1)!^2}{j\sqrt{2j+1}} \le M^{j-1}\frac{j!^2}{(j+1)^2}.
\end{aligned}$$

Hence, we can use the lemmas we derived in Faà di Bruno section and suppose $C_1 \geq 16$:

$$\begin{aligned}
&\| \sum_{l=2}^{k} C_l^k H^{(l)}(\gamma)\{(\frac{d}{dh})^{m-l}[\int_0^1 \gamma'(s+h\theta)d\theta]^l\}|_{h=0}\|_{L^2[0,s]} \\
&\leq \sum_{l=2}^{k} C_l^k \|H^{(l)}\|_{L^2[0,s]} \cdot \|\{(\frac{d}{dh})^{k-l}[\int_0^1 \gamma'(s+h\theta)d\theta]^l\}|_{h=0}\|_{L^2[0,s]}\|_{L^2[0,s]} \\
&\leq \sum_{l=2}^{k} C_l^k C_1\sqrt{s_0}(\frac{C}{M})^{l-1} M^{k-l-1}\frac{(k-l)!^2}{(k-l+1)^2} = \frac{C_1\sqrt{s_0}M^k}{C}\sum_{l=2}^{k}(\frac{C}{M^2})^l \frac{k!(k-l)!}{l!(k-l+1)^2} \\
&\leq \frac{C_1\sqrt{s_0}M^k}{C}\sum_{l=2}^{k}(\frac{C}{M^2})^2\frac{k!(k-l)!}{2!(k-l+1)^2} \leq 16C_1C\sqrt{s_0}M^{k-4}(k-2)!^2.
\end{aligned}$$

Finally, the desired inequality is obtained by triangle inquality.

LEMMA 3. *Assume the induction hypothesis for $j = 1, 2, \cdots, k$, we have*

$$\begin{aligned}
|I_2| &= |4\sum_{l=1}^{k-2} C_l^k \int_0^s \gamma^{(l+1)}(t^2\gamma^{(l+2)} + (2l+1)t\gamma^{(l+1)} + l^2\gamma^{(l)})(\frac{d}{dt})^{k-l}\eta(t\gamma')dt| \\
&\leq 4C\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}M^{k-2}\frac{k!(k-1)!}{(k+1)^2}
\end{aligned}$$

Proof. *By induction hypothesis, Schwarz inequality, M is large, s_0 is small, and (3.4.3),*

$$\begin{aligned}
|I_2| &\leq 4\sum_{l=1}^{k-2} C_l^k \int_0^s |\gamma^{(k+1)}| \cdot [s_0M^{l-1}\frac{l!^2}{l+1} + lM^{l-2}\frac{(l-1)!^2}{l}] \cdot 16M^{k-l-2}\frac{(k-l-1)!^2}{k-l}dt \\
&\leq 4\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]} \cdot 16M^{k-3}[s_0 + \frac{2}{M}]\sum_{l=1}^{k-2}\frac{k!l!(k-l-1)!}{(k-l)^2(l+1)} \\
&\leq 64\sqrt{s_0}M^{k-3}\|\gamma^{(k+1)}\|_{L^2[0,s]}[s_0 + \frac{2}{M}]\frac{k!(k-1)!}{(k+1)^2}[\frac{(k+1)^2}{(k-1)!}\sum_{l=1}^{k-2}\frac{(l+1)!(k-l-1)!}{(k-l)^2(l+1)^2}] \\
&\leq 512\sqrt{s_0}M^{k-3}\|\gamma^{(k+1)}\|_{L^2[0,s]}[s_0 + \frac{2}{M}]\frac{k!(k-1)!}{(k+1)^2} \\
&\leq C\sqrt{s_0}M^{k-2}\|\gamma^{(k+1)}\|_{L^2[0,s]}\frac{k!(k-1)!}{(k+1)^2}.
\end{aligned}$$

LEMMA 4. *Assume the induction hypothesis for $j = 1, 2, \cdots, k$, there is a constant C depending on H such that*

$$\begin{aligned}
|I_3| &= |4\int_0^s \gamma^{(k+1)}(t^2\gamma'' + t\gamma')\{(\frac{d}{dt})^k\eta(t\gamma') + 8t\gamma'\eta^2(t\gamma')[t\gamma^{(k+1)} + k\gamma^{(k)}]\}dt| \\
&\leq 4C\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}M^{k-4}\frac{k!(k-2)!}{(k-1)^2}.
\end{aligned}$$

Proof. *By s_0 is small, M is large, and (3.4.4):*

$$
\begin{aligned}
|I_2| &\leq 4CM^{k-4}\frac{k!(k-2)!}{(k-1)^2}\int_0^s |\gamma^{(k+1)}[t\frac{d}{dt}(t\gamma')]|dt \\
&\leq 4CM^{k-4}\frac{k!(k-2)!}{(k-1)^2}\frac{s_0}{M}\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]} \\
&\leq 4C\sqrt{s_0}\|\gamma^{(k+1)}\|_{L^2[0,s]}M^{k-4}\frac{k!(k-2)!}{(k-1)^2}.
\end{aligned}
$$

LEMMA 5. *Assume the induction hypothesis for $j = 1, 2, \cdots, k$, there is a constant C depending on H such that*

$$
\begin{aligned}
|I_4| = &|4k(k-1)^2\int_0^s \frac{d}{dt}\eta(t\gamma)\gamma^{(k-1)}\gamma^{(k+1)}dt - 32k\int_0^s (t^2\gamma'' + t\gamma')t\gamma'\eta^2(t\gamma')\gamma^{(k)}\gamma^{(k+1)}dt \\
&+ \int_0^s [4k^2\eta(t\gamma') + 4k(2k-1)t\frac{d}{dt}\eta(t\gamma') - 2nk]\gamma^{(k)}\gamma^{(k+1)}dt| \\
&\leq 16CM^{k-3}\frac{(k-1)!^2}{k}\|\gamma^{(k+1)}\|_{L^2[0,s]}.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
|I_4| &\leq C\|\gamma^{(k+1)}\|_{L^2[0,s]}(k^2\|\gamma^{(k)}\|_{L^2[0,s]} + k^3\|\gamma^{(k-1)}\|_{L^2[0,s]}) \\
&\leq C\|\gamma^{(k+1)}\|_{L^2[0,s]}(k^2M^{k-3}\frac{(k-2)!^2}{k-1} + k^3M^{k-4}\frac{(k-3)!^2}{k-2}) \\
&\leq C\|\gamma^{(k+1)}\|_{L^2[0,s]}M^{k-3}\frac{(k-1)!^2}{k}[\frac{k^3}{(k-1)^3} + \frac{1}{M}\frac{k^4}{(k-2)^3(k-1)^2}] \\
&\leq 16C\|\gamma^{(k+1)}\|_{L^2[0,s]}M^{k-3}\frac{(k-1)!^2}{k}.
\end{aligned}
$$

Now, the result can be proved in the same way as we derive the apriori bound of Theorem 3.

Proof.

$$
\begin{aligned}
&\|\gamma^{(k+1)}\|_{L^2[0,s]} + |(\frac{d}{ds})^k[s\gamma'(s)]| \\
&\leq C\sqrt{s_0}\{M^{k-2}\frac{(k-1)!k!}{(k+1)^2} + M^{k-4}\frac{k!(k-2)!}{(k-1)^2}\} \\
&+ \frac{C}{M}M^{k-2}\frac{(k-1)!^2}{k} + C_2M^{k-3}(k-2)!^2 \\
&\leq M^{k-2}\frac{(k-1)!^2}{k}(C\sqrt{s_0} + C\sqrt{s_0}\frac{1}{M^2}) \\
&\leq C\sqrt{s_0}M^{k-2}\frac{(k-1)!^2}{k} \leq M^{k-2}\frac{(k-1)!^2}{k}
\end{aligned}
$$

for s_0 small and M large.

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Curriculum Vitae

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